JOURNAL OF APPROXIMATION THEORY 57, 300–305 (1989)

Quadratic Spline Interpolation

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Communicated by G. Meinardus

Received July 14, 1987

1. INTRODUCTION

Consider a mesh P of $[0, 1]$ given by $0 = x_0 < x_1 < \cdots < x_n = 1$ such that $x_i - x_{i-1} = h$, for $i = 1, 2, ..., n$. Let π_m denote the set of all algebraic polynomials of degree m or less. For a function s defined over $[0, 1]$, we denote the restriction of s over $[x_{i-1}, x_i]$ by s_i. The class of periodic quadratic splines over $[0, 1]$ with mesh P is defined by

$$S(2, P) = \{s : s_i \in \pi_2, s \in C^1[0, 1], s^{(j)}(0) = s^{(j)}(1), j = 0, 1\}.$$

Assuming that f is a 1-periodic locally integrable function with respect to a nonnegative real measure $d\mu$, Sharma and Tzimbalario [5] have proved the following for quadratic splines:

THEOREM 1. *Suppose for some i,*

$$\int_{x_{i-1}^+}^{x_i^-} d\mu > 0 \textit{ and for every } j, \int_{x_{j-1}}^{x_j} d\mu > 0.$$

Then there exists a unique 1-periodic $s \in S(2, P)$ which satisfies the interpolatory condition,

$$\int_{x_{i-1}}^{x_i} (f(x) - s(x))\, d\mu = 0, \qquad i = 1, 2, ..., n. \tag{1.1}$$

It may be mentioned that condition (1.1) reduces to different interpolatory conditions by suitable choice of $\mu(x)$. Thus, if μ has a jump of 1 at $h/2$ then condition (1.1) becomes the interpolatory condition,

$$s(u_i) = f(u_i); \qquad u_i = (x_{i-1} + x_i)/2; \qquad i = 1, 2, ..., n. \tag{1.2}$$

The error bounds for quadratic splines which interpolate a given func-

0021-9045/89 $3.00

tion at the midpoint of every mesh have been obtained by Marsden [3]. Considering $f \in C^4$ Rosenblatt [4] has obtained asymptotically precise estimates for the derivate of the difference between the cubic spline interpolating at mesh points and the function interpolated. In the present paper, we obtain a similar estimate for the quadratic spline interpolating at the midpoints between the successive mesh points. It may be worthwhile to mention that Boneva *et al.* [2] have shown the use of the derivative of a cubic spline interpolator for smoothing of histograms.

Without any loss of generality, we assume for the rest of this paper that the quadratic spline s under consideration satisfies the condition $s'(0) = 0$. Thus, it follows from the proof of Theorem 1 that in the case of midpoint interpolation over a uniform mesh the system of equations for determining the first derivative $m_i = s'(x_i)$ of the quadratic spline interpolant s is written as

$$(m_{i-1} + 6m_i + m_{i+1})/2 = F_i, \qquad i = 1, 2, ..., n-1, \tag{1.3}$$

where $F_i = 4\{f(u_{i+1}) - f(u_i)\}/h$.

2. Estimation of the Inverse of the Coefficient Matrix

Denoting the transposes of $[m_1, m_2, ..., m_{n-1}]$ and $[F_1, F_2, ..., F_{n-1}]$ by M and F, respectively, it may be observed that the system of equations (1.3) is written as

$$AM = F, \tag{2.1}$$

where A is the square matrix of order $n-1$ coefficient matrix. A is of course invertible (see [5, p. 188]).

Following Ahlberg *et al.* [1] who have estimated the inverse of the coefficient matrix for the case of the cubic spline matching the function at mesh points of uniform mesh, we estimate the inverse of A. For this we introduce the following square matrix of order n

$$D_n(\alpha, \beta) = \begin{bmatrix} 2\beta & \alpha & 0 & . & . & . & 0 & 0 & 0 \\ 1-\alpha & 2\beta & \alpha & . & . & . & 0 & 0 & 0 \\ 0 & 1-\alpha & 2\beta & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1-\alpha & 2\beta & \alpha \\ 0 & 0 & 0 & . & . & . & 0 & 1-\alpha & 2\beta \end{bmatrix},$$

where α and β are given real numbers such that $\beta \geqslant \alpha$. By using the induction hypothesis, it is easily seen that $|D_n|$ satisfies the difference equations:

$$|D_n(\alpha, \beta)| - 2\beta\, |D_{n-1}(\alpha, \beta)| + \alpha(1-\alpha)\, |D_{n-2}(\alpha, \beta)| = 0 \tag{2.2}$$

with $|D_{-1}(\alpha, \beta)| = 0$, $|D_0(\alpha, \beta)| = 1$, $|D_1(\alpha, \beta)| = 2\beta$, and

$$2\eta\, |D_n(\alpha, \beta)| = (\beta+\eta)^{n+1} - (\beta-\eta)^{n+1} \tag{2.3}$$

with

$$\eta = (\beta^2 - \alpha + \alpha^2)^{1/2}.$$

Further, it may also be observed that for $\alpha = \frac{1}{2}$,

$$b^{-n}(2\beta + r)\, |D_n(\alpha, \beta)| = 2\beta(1 - r^{2n}) + r(1 - r^{2n-2})/2, \tag{2.4}$$

where $r = (-1/2b) = -2[\beta - (\beta^2 - \frac{1}{4})^{1/2}]$.

Taking $2\beta = 3$ and $\alpha = \frac{1}{2}$ in $D_n(\alpha, \beta)$, we observe that the coefficient matrix A satisfies the following difference equation:

$$4\, |A| = 12\, |D_{n-2}(\tfrac{1}{2}, \tfrac{3}{2})| - |D_{n-3}(\tfrac{1}{2}, \tfrac{3}{2})|. \tag{2.5}$$

Thus, it follows from (2.4) that

$$(3+r)\, b^{2-n}\, |A| = (3 + r/2)^2 - r^{2n-6}(3r + 1/2)^2. \tag{2.6}$$

We can get the elements $a_{i,j}$ of A^{-1} from the cofactors of the transpose matrix. Thus, for $0 < i \leqslant j \leqslant n-2$ or $i = j = 0$ (cf. [1, pp. 35–38]),

$$|A|\, a_{i,j} = (b \cdot r)^{j-i}\, D_i(\tfrac{1}{2}, \tfrac{3}{2})\, D_{n-j-2}(\tfrac{1}{2}, \tfrac{3}{2})$$

and

$$|A|\, a_{0,j} = (b \cdot r)^j\, D_{n-j-2}(\tfrac{1}{2}, \tfrac{3}{2}) \qquad \text{for} \quad 0 < j \leqslant n-2.$$

Thus, using (2.5) and (2.4), we see that for $0 \leqslant i \leqslant j \leqslant n-2$

$$(1 - r^2)(1 - r^{2n})a_{i,j} = -2r^{j+1-i}(1 - r^{2i+2})(1 - r^{2n-2j-2}).$$

From the above expression, we see that A^{-1} is symmetric. Now considering a fixed value x such that $0 < x < 1$, we observe that for fixed $\varepsilon > 0$ and $\varepsilon < i/n$, $j/n < 1 - \varepsilon$ the elements $a_{i,j}$ of A^{-1} may be approximated asymptotically by $r^{|j-i|}(3+r)^{-1}$.

We thus prove the following:

THEOREM 2. *The coefficient matrix A of* (2.1) *is invertible and the* *elements* $a_{i,j}$ *of* A^{-1} *can be approximated asymptotically by* $r^{|j-i|}(3+r)^{-1}$ *and the row* max *norm of its inverse*; *that is*,

$$\|A^{-1}\| \leqslant \frac{(1+r)}{(1-r)(3+r)}, \tag{2.7}$$

where $r=\sqrt{8}-3$.

Remark 2.1. It is interesting in note that the estimate (2.7) is sharper than that obtained in terms of the infimum of the excess of the positive value of the leading diagonal element over the sum of the positive values of the other elements in each row. For the latter of these gives $\|A^{-1}\| \leqslant 0.5$ whereas (2.7) shows that the $\|A^{-1}\|$ does not exceed 0.25.

Since A is invertible, it follows from the proof of Theorem 1 that there exists a unique spline $s \in S(2, P)$ which satisfies the interpolatory condition (1.2).

3. Error Bounds

Considering throughout this section a 1-periodic function $f \in C^3$, we shall estimate the error function $e=s-f$, where s is the quadratic spline interpolant of f satisfying the interpolatory condition (1.2). Considering the interval $[x_{i-1}, x_i]$, we observe that, since s' is linear in the interval $[x_{i-1}, x_i]$,

$$hs'(x)=(x-x_{i-1})\, m_i+(x_i-x)\, m_{i-1}. \tag{3.1}$$

Thus, we get

$$2hs(x)=(x-x_{i-1})^2 m_i-(x_i-x)^2 m_{i-1}+2hc_i, \tag{3.2}$$

where the constant c_i is to be determined from the interpolatory condition (1.2). Hence,

$$8f(u_i)=h(m_i-m_{i-1})+8c_i. \tag{3.3}$$

Using (3.3) in (3.2), we have

$$8hs(x)=[4(x-x_{i-1})^2-h^2]m_i+[h^2-4(x_i-x)^2]m_{i-1}+8hf(u_i). \tag{3.4}$$

Replacing m_i by $e'(x_i)$ in (3.4), we notice that

$$8hs(x)=[4(x-x_{i-1})^2-h^2]\, e'(x_i)+[h^2-4(x_i-x)^2]\, e'(x_{i-1})+G_i(f), \tag{3.5}$$

where $G_i(f)=[4(x-x_{i-1})^2-h^2]f'(x_i)+[h^2-4(x_i-x)^2]f'(x_{i-1})+8hf(u_i)$. We see that $G_i(f)$ may be expressed as a linear combination of the values of the third derivative f''' of f. For since $f\in C^3$, we have by Taylor's theorem,

$$G_i(f)=8hf(x)+f'''(x)(x_i+x_{i-1}-2x)[(x_i+x_{i-1}-2x)^2-3h^2]h/6+o(h^4),$$

where x is an appropriate point in (x_{i-1}, x_i) which is not necessarily the same at each occurrence. Writing Eq. (2.1) as

$$A(e'(x_i))=(F_i)-A(f'(x_i))=(D_i), \tag{3.6}$$

say, we first estimate (D_i).

Applying Taylor's theorem to the right-hand side of (3.6) we have

$$(D_i)=-h^2f'''(x)/3+O(h^2). \tag{3.7}$$

Taking a sufficiently large but fixed positive integer m and noticing that $A^{-1}=(a_{i,j})$, we have from Eq. (3.6)

$$(e'(x_i))=\left(\sum_{|k-i|\geqslant m}+\sum_{|k-i|<m}\right)(a_{i,k}D_k)=(R_1)+(R_2)$$

say. We shall estimate R_1 and R_2 separately. Suppose that x is a fixed point in (0, 1) and let $x_i=[nx]/n$ where $[y]$ is the greatest integer less than or equal to y. Then, it is clear that as $n\to\infty$ $i\cong nx$ and $n-i\cong n(1-x)$. Now assuming that f''' is monotonic, we see from Theorem 2 that

$$|(R_1)|\leqslant k_1(0.25)^m h^2, \tag{3.8}$$

where k_1 is an absolute constant.

Next, we observe that the points x_k for the values of k occurring in R_2 satisfy

$$|x_k-x|=O(h). \tag{3.9}$$

Thus, using the result of Theorem 2, (3.7), and the continuous differentiability of f''', we have

$$\left|(R_2)-\sum_{|k-i|<m}\frac{r^{|k-i|}}{(3+r)}(-h^2f'''(x)/3)\right|=O(h^2). \tag{3.10}$$

Since m is arbitrary, we get

$$(e'(x_i))=-(h^2/12)f'''(x)+O(h^2). \tag{3.11}$$

Combining (3.11) with (3.5), we prove the following:

THEOREM 3. *Let s be the quadratic spline interpolant of a* 1*-periodic function f satisfying* (1.2). *Let* f''' *exist and be a nonnegative monotonic continuous function. Then for any fixed point x such that* $0 < x < 1$,

$$e(x) = f'''(x)(x_i + x_{i-1} - 2x)[(x_i + x_{i-1} - 2x)^2 - h^2]/48 + O(h^3) \tag{3.12}$$

as $n \to \infty$.

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